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TWO-DIMENSIONAL CONTACT ON A PIEZOELECTRIC HALF-SPACE

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Abstract—Due to their intrinsic electro-mechanical coupling effect, piezoelectric materials have been widely used in industry. In the present paper, stress and electrical field distributions in a piezoelectric half-plane under contact load at the surface are considered. Since a piezoelectric material is intrinsically anisotropic, stress analysis has been impeded by the complexity raised by too many material constants. Hereby, Stroh's formalism is applied in the present study to overcome this difficulty. The solution for a concentrated force and charge acting on the boundary of the half-space, the Green function, is obtained in a neat form. The non-slip and slip indentor contacts on the piezoelectric half-space are also formulated.

1. INTRODUCTION

A piezoelectric half-space under contact loading is considered in the present paper. In the following sections, we will focus on a linear piezoelectric material whose constitutive equation is given by:

$$\sigma = C\gamma - eE, \quad D = e\gamma + \varepsilon E, \tag{1}$$

where $\mathbf{C}$ is the elasticity tensor of rank four, ε the permittivity tensor of rank two and $\mathbf{e}$ the piezoelectricity tensor of rank three. When the piezoelectricity vanishes, the problem decouples into an anisotropic elastic and a dielectric problem.

Due to their anisotropic behavior, piezoelectric materials are described by many material constants. The stress and electric field analyses of piezoelectric solids have been impeded by the difficulties raised by these material constants. For instance, results for piezoelectric plate vibration (Tiersten, 1969) showed great complexity because of the anisotropic behavior of the material. Nevertheless, a satisfactory formulation for two-dimensional anisotropic elasticity has been developed via the Stroh (1958) formalism. This formulation has been proved to be elegant and powerful for studies of dislocation (Stroh, 1958), wave propagation (Barnett and Lothe, 1985) and interfacial cracks (Ting, 1986). Barnett and Lothe (1975) applied this formalism to a piezoelectric material when a dislocation in an infinite piezoelectric medium was studied. More recent works by Suo *et al.* (1992) for an interfacial crack in a piezoelectric composite and by Fan (1995) for the Saint-Venant end effect in a piezoelectric strip have both benefitted from the Stroh formalism. This formalism is applied again here for piezoelectric contact problems.

2. STROH'S FORMULATION FOR PIEZOELECTRIC MATERIALS

In rectangular coordinates the linear piezoelectric solid is described by:
Constitutive laws:

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\gamma_{kl} - e_{kij}E_k \\ D_i &= e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k \end{aligned} \tag{2}$$

where σ_{ij}, γ_{ij}, D_i and E_k are stress, strain, electric displacement (or electric induction) and electric field, respectively.
Deformation relations:

$$\begin{aligned} \gamma_{kl} &= \tfrac{1}{2}(u_{k,l} + u_{l,k}), \\ E_k &= -\varphi_{,k}, \end{aligned} \tag{3}$$

where u_k and φ are mechanical displacement and electric potential, respectively.
Equilibrium equations:

$$\begin{aligned} \sigma_{ij,i} &= 0, \\ D_{i,i} &= 0, \end{aligned} \tag{4}$$

provided body force and electric source are absent. Substituting eqns (2) and (3) into eqn (4) yields:

$$\begin{aligned} (C_{ijkl}u_k + e_{lji}\varphi)_{,li} &= 0 \\ (e_{ikl}u_k - \varepsilon_{il}\varphi)_{,li} &= 0. \end{aligned} \tag{5}$$

If all the fields are independent of the third coordinate, say x_3, special solutions can be sought in the form:

$$\mathbf{U} = \{u_k, \varphi\}^{\mathrm{T}} = f(\zeta_1 x_1 + \zeta_2 x_2)\mathbf{a}, \tag{6}$$

where, without loss of generality,

$$\zeta_1 = 1, \quad \zeta_2 = p \tag{7}$$

and $\mathbf{a} = (a_1, a_2, a_3, a_4)^{\mathrm{T}}$ is independent of the spatial coordinates.

A direct substitution of eqn (6) into eqn (7) gives

$$\begin{aligned} (C_{\alpha ik\beta}a_k + e_{\alpha i\beta}a_4)\zeta_\alpha\zeta_\beta &= 0 \\ (e_{\alpha k\beta}a_k - \varepsilon_{\alpha\beta}a_4)\zeta_\alpha\zeta_\beta &= 0. \end{aligned} \tag{8}$$

For non-zero values of $\mathbf{a}$, we must have:

$$\det \begin{bmatrix} C_{\alpha jk\beta}\zeta_\alpha\zeta_\beta & e_{\alpha j\beta}\zeta_\alpha\zeta_\beta \\ e_{\alpha k\beta}\zeta_\alpha\zeta_\beta & -\varepsilon_{\alpha\beta}\zeta_\alpha\zeta_\beta \end{bmatrix} = 0. \tag{9}$$

This is a nonlinear eigenvalue problem. However, it can be converted to a standard linear eigenvalue problem via the so-called eight-dimensional representation, as summarized in Appendix A.

As in the anisotropic elasticity formulation, it can be proved that the eigenvalue p cannot be purely real due to the positive definiteness of the tensors C_{ijkl} and ε_{ij}. Four pairs of p can be arranged as:

$$p_{I+4} = \bar{p}_I, \quad (I = 1, 2, 3 \text{ and } 4)$$

$$p_I = \alpha_I + i\beta_I, \quad \beta_I > 0. \tag{10}$$

Corresponding to the eigenvalues $p_I = \alpha_I + i\beta_I$, there are four independent eigenvectors which form a 4×4 matrix:

$$\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}. \tag{11}$$

The complex conjugates

$$\bar{\mathbf{A}} = \{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3, \bar{\mathbf{a}}_4\} \tag{12}$$

are the eigenvectors corresponding to $p_{I+4} = \bar{p}_I$.

Using the eigenvalues and corresponding eigenvectors, the general solution can be written as a linear combination of the eight eigenvectors:

$$\mathbf{U} = 2\,\mathrm{Re}\left[(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \begin{pmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{pmatrix} \right], \tag{13}$$

where

$$z_I = x_1 + p_I x_2.$$

For the sake of convenience, we write this in a compact form,

$$\mathbf{U} = 2\,\mathrm{Re}\,[\mathbf{Af}(z)]. \tag{13'}$$

Conversion from this compact form to a four-complex-variable form is demonstrated in Appendix B.

Furthermore, from the constitutive equations, we have

$$\mathbf{t} = \{\sigma_{2i}, D_2\} = 2\,\mathrm{Re}\,[\mathbf{Bf}'(z)] \tag{14}$$

$$\mathbf{s} = \{\sigma_{1i}, D_1\} = -2\,\mathrm{Re}\,[\mathbf{BPf}'(z)], \tag{15}$$

where

$$\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}, \quad \mathbf{P} = \mathrm{diag}\,\{p_1, p_2, p_3, p_4\} \tag{16}$$

$$\mathbf{b}_j = (C_{2jk\beta} a_k + e_{\beta j2} a_4)\zeta_\beta,$$

$$\mathbf{b}_4 = (-\varepsilon_{1\beta} a_4 + e_{2k\beta} a_k)\zeta_\beta. \tag{17}$$

It has been noted that the matrices **A** and **B** are non-singular when the eigenvalues are distinct. However, they may be singular in some special cases in which eigenvalues coincide. Ting and co-workers have undertaken extensive studies on these degenerate cases in anisotropic elasticity. A similar theorem and discussions are expected for piezoelectricity. In the following sections, we will take **A** and **B** as non-singular matrices. The degenerate cases where **A** and **B** are singular will be discussed elsewhere.

3. MIXED BOUNDARY VALUE PROBLEM FOR A PIEZOELECTRIC HALF SPACE

Piezoelectric material is taken to occupy the region $x_2 > 0$, whose boundary, the x_1-axis, is divided into two parts, namely the contact region ($\forall x_1 \in (-a, a)$) and its complement. In the contact region, the traction, electric potential and induction can be written as

$$\begin{aligned} \sigma_{2j} &= \sigma_{2j}^0 \\ \varphi &= \varphi^0 \\ D_2 &= D_2^0. \end{aligned} \tag{18}$$

The right hand side terms in the equation are the values in the indentor, which may or may not be known before we solve the problem. The boundary condition on the latter part is assumed to be

$$\begin{aligned} \sigma_{2i}(x_1, 0) &= 0 \\ D_2(x_1, 0) &= 0 \quad x_1 \notin (-a, a). \end{aligned} \tag{19}$$

Boundary conditions other than eqns (19), for instance a prescribed φ, can also be considered by a formulation provided by Suo (1993). In the present section, we focus on eqns (19) to demonstrate the approach.

With eqns (18) and (19) and the notation of eqn (14), we may write

$$\mathbf{Bf}'(x_1) + \overline{\mathbf{Bf}'(x_1)} = \mathbf{t}(x_1) \quad \forall x_1 \in (-\infty, \infty). \tag{20}$$

For later convenience, we introduce a function

$$\begin{aligned} \mathbf{h}(z) &= \mathbf{Bf}'(z) \quad \forall x_2 > 0 \\ \mathbf{h}(z) &= -\overline{\mathbf{Bf}'(\bar{z})} \quad \forall x_2 < 0, \end{aligned} \tag{21a,b}$$

which is analytic throughout the whole plane except on the contact segment. Using limits of the function $\mathbf{h(z)}$ on $x_2 = 0$ (England, 1971), eqn (20) leads to a Hilbert problem

$$\begin{aligned} \mathbf{h}^+(x_1) - \mathbf{h}^-(x_1) &= \mathbf{t}(x_1), \quad \forall x_1 \in (-a, a) \\ \mathbf{h}^+(x_1) - \mathbf{h}^-(x_1) &= 0, \qquad \forall x_1 \notin (-a, a). \end{aligned} \tag{22}$$

The solution of eqn (22) can be obtained as:

$$\mathbf{h}(z) = \frac{1}{2\pi i} \int_{-a}^{a} \frac{\mathbf{t}(x)}{x - z} \mathrm{d}x, \tag{23}$$

if the distributions of the traction and electric induction are known over the region $\forall x_1 \in (-a, a)$. As a particular example, let us take the traction and electric induction to be uniform over this region. Then

$$\mathbf{h}(z) = \frac{\mathbf{t}}{2\pi i} \ln\left(\frac{z - a}{z + a}\right). \tag{24}$$

If we take:

$$\lim_{a\to 0} 2a\mathbf{t} = \mathbf{T}, \tag{25}$$

we have the two-dimensional Green function for the piezoelectric half space as:

$$\mathbf{h}(z) = -\frac{\mathbf{T}}{2\pi iz}. \tag{26}$$

This one-complex-variable solution, eqn (26), will be converted to a four-complex-variable solution in Appendix B.

A similar problem has been considered by Sosa and Castro (1994) for a simplified constitutive equation.

3.1. *Non-slip contact*

When the traction–induction is unknown in the contact region, while all the displacement components and electric potential are prescribed in the contact region, the solution is obtained by considering a Hilbert problem. Using the notation of eqns (12) and (13), the displacement–potential continuity across the interface reads as

$$\mathbf{A}\mathbf{f}'(x_1)+\overline{\mathbf{A}\mathbf{f}'(x_1)} = \mathbf{d}'(x_1) \quad \forall x_1 \in(-a,a) \tag{27}$$

and the traction–induction free condition outside the contact zone leads to:

$$\mathbf{B}\mathbf{f}'(x_1)+\overline{\mathbf{B}\mathbf{f}'(x_1)} = 0 \quad \forall x_1 \notin(-a,a), \tag{28}$$

where

$$\mathbf{d} = \{u_1, u_2, u_3, \varphi\}^{\mathrm{T}} \tag{29}$$

is assumed to be known inside the contact region. By using the function defined in eqn (21), eqn (27) is rewritten as:

$$\mathbf{h}^+(x_1)+\mathbf{Y}^{-1}\bar{\mathbf{Y}}\mathbf{h}^-(x_1) = i\mathbf{Y}^{-1}\mathbf{d}'(x_1), \quad \forall x_1 \in(-a,a), \tag{30}$$

where

$$\mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1}. \tag{31}$$

The matrix $\mathbf{Y}$ has been discussed by Suo *et al.* (1992). The properties of this matrix will be mentioned when it is used later on. If the matrix is real, the solution of eqn (30) is given by (England, 1971):

$$\mathbf{h}(z) = \frac{\chi(z)}{2\pi}\int_{-a}^{a}\frac{\mathbf{Y}^{-1}\mathbf{d}'(x)}{\chi^+(x)(x-z)}\mathrm{d}x+\chi(z)\mathbf{Q}(z), \tag{32}$$

where

$$\chi(z) = ((z-a)(z+a))^{-1/2}$$

and $\mathbf{Q}$ is a polynomial to be determined by considering the resultant force acting on the contact zone (Fan and Keer, 1994).

In general, $\mathbf{Y}$ is a complex matrix. The Hilbert problem, eqn (30), is solved only if we can transform the matrix $\mathbf{Y}^{-1}\bar{\mathbf{Y}}$ in eqn (30) into a diagonal form. Following the approach proposed by Ting (1986) for anisotropic elasticity, Suo *et al.* (1992) modified the transformation procedure for piezoelectricity. By considering an eigenvalue problem as

$$\bar{\mathbf{Y}}\mathbf{w} = e^{2\pi\lambda}\mathbf{Y}\mathbf{w}, \tag{33}$$

we have four eigenpairs (eigenvalue, eigenvector) as:

$$(\varepsilon, \mathbf{w}), \quad (-\varepsilon, \bar{\mathbf{w}}), \quad (-i\kappa, \mathbf{w}_3), \quad (i\kappa, \mathbf{w}_4). \tag{34}$$

Any field can be decomposed via this eigensystem, say,

$$\begin{aligned} \mathbf{h} &= h_1\mathbf{w}+h_2\bar{\mathbf{w}}+h_3\mathbf{w}_3+h_4\mathbf{w}_4 \\ \mathbf{d} &= d_1\mathbf{w}+d_2\bar{\mathbf{w}}+d_3\mathbf{w}_3+d_4\mathbf{w}_4. \end{aligned} \tag{35}$$

With the eigen-expansion eqn (35), we can decouple eqn (30) as:

$$\begin{aligned} h_1^+ + e^{-2\pi\varepsilon}h_1^- &= i\hat{d}'_1 \\ h_2^+ + e^{-2\pi\varepsilon}h_2^- &= i\hat{d}'_2 \\ h_3^+ + e^{2\pi i\kappa}h_3^- &= i\hat{d}'_3 \\ h_4^+ + e^{-2\pi i\kappa}h_4^- &= i\hat{d}'_4, \end{aligned} \tag{36}$$

where

$$\hat{\mathbf{d}} = \mathbf{Y}^{-1}\mathbf{d}.$$

The solutions are obtained in a form like eqn (32).

It is worthwhile mentioning that there are two kinds of indentor, namely, a sharp edged and a round edged indentor. For the sharp edged one, the contact zone is known. At the two corners, the stress is singular by analogy with isotropic elastic contact [see Johnson (1985)]. However, the singularities for this piezoelectric contact problem are more complicated than those in elasticity. Fortunately, the detailed structure of the singularities has been discussed by Suo *et al.* (1992) when they studied the interfacial crack in piezoelectric bimaterials. In the case of a round edged indentor, there is no singularity in the solution. The contact zone size is determined by considering the resultant force acting on the half-plane.

3.2. *Slip contact*

If the interfacial static friction of the contact is not high enough to sustain the slip between the two bodies, some displacement components in the x-direction may be discontinuous. Thus, the boundary condition eqn (27) must be modified as, for instance,

$$u_2 = \bar{u}_2 \quad \text{and} \quad \sigma_{12} = \mu\sigma_{22}, \tag{37}$$

together with proper electrical conditions. In eqn (37), μ is the sliding friction coefficient.

Let us summarize the displacement–potential and traction–induction conditions inside the contact region:

$$\mathbf{Y}\mathbf{h}^+(x_1)+\bar{\mathbf{Y}}\mathbf{h}^-(x_1) = i\mathbf{d}'(x_1), \quad \forall x_1 \in (-a, a) \tag{38}$$

$$\mathbf{h}^+(x_1)-\mathbf{h}^-(x_1) = \mathbf{t}(x_1), \quad \forall x_1 \in (-a, a). \tag{39}$$

Eliminating $\mathbf{h}^-(x)$ from eqn (38), it follows that:

$$(\mathbf{Y}+\bar{\mathbf{Y}})\mathbf{h}^+(x_1)-\bar{\mathbf{Y}}\mathbf{t}(x_1) = i\mathbf{d}'(x_1). \tag{40}$$

On the other hand, the Plemelj formula gives:

$$\mathbf{h}^+(x_1) = \frac{1}{2}\mathbf{t}(x_1) + \frac{1}{2\pi i}\int_{-a}^{a}\frac{\mathbf{t}(\xi)}{\xi - x_1}\mathrm{d}\xi. \tag{41}$$

Substituting eqn (41) into eqn (40), one has

$$\frac{\mathbf{Y}-\bar{\mathbf{Y}}}{2}\mathbf{t}(x_1) + \frac{\mathbf{Y}+\bar{\mathbf{Y}}}{2\pi i}\int_{-a}^{a}\frac{\mathbf{t}(\xi)}{\xi - x_1}\mathrm{d}\xi = i\mathbf{d}'(x_1). \tag{42}$$

There are four equations for four unknowns in eqn (42). Thus, eqn (42) provides the distribution of traction inside the contact region, which in turn allows us to have the whole field solution via eqn (23).

4. SPECIAL CASES

4.1. *Decoupled elastic and dielectric contact problems*

When the piezoelectric tensor vanishes, the problem is decoupled into anisotropic elastic and dielectric ones. The anisotropic elastic contact problem has been considered by Fan and Keer (1994) via Stroh's notation. The latter, a mixed boundary value problem for a general anisotropic dielectric half-space, is presented here.

In the case $\mathbf{e} = \mathbf{0}$, the eigenvalue problem eqn (8) is simplified as

$$C_{\alpha ik\beta}\zeta_\alpha\zeta_\beta a_k = 0 \tag{43a}$$

and

$$\varepsilon_{\alpha\beta}\zeta_\alpha\zeta_\beta a_4 = 0. \tag{43b}$$

The corresponding eigenvectors are in the form of:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_e & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_e & 0 \\ 0 & b_4 \end{pmatrix}, \tag{44}$$

where $\mathbf{A}_e$ and $\mathbf{B}_e$ are 3×3 matrices corresponding to anisotropic elasticity. The dielectric problem is formed by eqn (43b) and scalars in eqn (44). The eigenvalues for the anisotropic dielectricity are obtained from eqn (43b):

$$\varepsilon_{11} + 2\varepsilon_{12}p + \varepsilon_{22}p^2 = 0. \tag{45}$$

They are:

$$p_4 = -\frac{\varepsilon_{12}}{\varepsilon_{22}} + i\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{22}} - \left(\frac{\varepsilon_{12}}{\varepsilon_{22}}\right)^2}, \quad p_8 = \bar{p}_4. \tag{46}$$

The last equation of (27), corresponding to dielectricity, is decoupled from the first three equations which are associated with anisotropic elasticity. Thus,

$$f'_4(x) + \bar{f}'_4(x) = \varphi'_0(x). \tag{47}$$

The solution of eqn (47) is easily obtained by applying eqn (32).

It is noted that the degree of electromechanical coupling in a piezoelectric material can be described by a dimensionless parameter formed by the three types of moduli, which is roughly in the range:

$$\frac{e}{\sqrt{\varepsilon C}} = 0.1 \sim 1. \tag{48}$$

A weakly coupled material such as quartz, which is widely used in frequency filters and resonators, has moduli of the order of (Salt, 1987):

$$C \sim 10^{11}\,\mathrm{N\,m^{-2}}, \quad \varepsilon \sim 10^{-11}\,\mathrm{F\,m^{-2}}, \quad e \sim 10^{-1}\,\mathrm{C\,m^{-2}}. \tag{49}$$

It is seen that:

$$\frac{e}{\sqrt{\varepsilon C}} \sim 0.1. \tag{50}$$

The mechanical and electric fields of this weakly coupled piezoelectric material can be approximated by the decoupled elastic and dielectric solutions.

On the other hand, a strongly electromechanical coupled material, such as lead–zirconate–titanate (say PZT-5H), has moduli of the order of:

$$C \sim 10^{11}\,\mathrm{N\,m^{-2}}, \quad \varepsilon \sim 10^{-8}\,\mathrm{F\,m^{-2}}, \quad e \sim 10\,\mathrm{C\,m^{-2}}. \tag{51}$$

The non-dimensional parameter is of the order of:

$$\frac{e}{\sqrt{\varepsilon C}} \sim 0.3. \tag{52}$$

For this kind of material, the mechanical and electric fields have to be calculated based upon the fully coupled formulation.

4.2. *Piezoelectric half-space with transverse symmetry around poling axis*

The above mentioned PZT-5H belongs to this material category. Assuming the x_1–x_2 plane is the isotropic plane, the material constants of this material are:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{13} \\ 2\gamma_{12} \end{bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{13} \\ 2\gamma_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \tag{53}$$

Let us consider a two-dimensional contact problem in the (x, y) plane; the in-plane deformation (u_x, u_y) is decoupled from the anti-plane field (u_z, φ). The former is identical to an elastic problem. We focus on the latter.

The eigenvalues corresponding to this problem are obtained from eqn (8) as

$$p_3 = p_4 = i, \quad \bar{p}_7 = \bar{p}_8 = -i \tag{54}$$

and the related matrices are

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{2e} & 0 \\ 0 & \mathbf{A}_p \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{2e} & 0 \\ 0 & \mathbf{B}_p \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{2e} & 0 \\ 0 & \mathbf{Y}_p \end{bmatrix}. \tag{55}$$

where the right upper 2×2 matrices correspond to in-plane deformation, while the left lower corner matrices correspond to the coupled anti-plane deformation and electric field. It is straightforward to derive:

$$\mathbf{A}_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_p = i \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix}, \tag{56}$$

and

$$\mathbf{Y}_p = \frac{1}{1+k} \begin{bmatrix} c_{44}^{-1} & ke_{15}^{-1} \\ ke_{15}^{-1} & -\varepsilon_{11}^{-1} \end{bmatrix}, \tag{57}$$

where

$$k = e_{15}^2/(c_{44}\varepsilon_{11}).$$

Since $\mathbf{Y}_p$ is a real matrix, then the solution, eqn (32), can be applied.

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APPENDIX A: EIGHT-DIMENSIONAL REPRESENTATION OF PIEZOELECTRICITY

In general, the eigenvalue problem eqn (8) requires a numerical procedure. A standard linear eigenvalue problem will allow us to use existing numerical subroutines. To convert eqn (8) into a standard linear eigenvalue problem, Barnett and Lothe (1975) introduced an eight-dimensional representation.

In the following formulation, lower case subscripts take on the range 1, 2 and 3, while the upper case subscripts take on the range 1, 2, 3 and 4. The eight-dimensional formulation introduces the following auxiliary symbols for matrix formulations.

$$Z_{Mn} = \begin{cases} \gamma_{mn} & M = 1,2,3 \\ -E_n & M = 4 \end{cases} \tag{A1}$$

$$\Sigma_{Mn} = \begin{cases} \sigma_{mn} & M = 1,2,3 \\ D_n & M = 4 \end{cases} \tag{A2}$$

$$U_M = \begin{cases} u_m & M = 1,2,3 \\ \varphi & M = 4 \end{cases} \tag{A3}$$

$$E_{iJMn} = \begin{cases} C_{ijmn} & J, M = 1,2,3 \\ e_{mi} & J = 1,2,3\,;\, M = 4 \\ e_{inm} & J = 4\,;\, M = 1,2,3 \\ -\varepsilon_{in} & J, M = 4. \end{cases} \tag{A4}$$

It should be pointed out that they are not tensors. One has to be careful when the coordinate system is changed.

In terms of the notation of eqns (A1)–(A4), the constitutive equations [eqns (2) and (3)] are written as;

$$\Sigma_{iJ} = E_{iJMn} Z_{Mn} = E_{iJMn} U_{M,n}. \tag{A5}$$

The equilibrium equation (4) is written as:

$$\Sigma_{iJ,i} = 0. \tag{A6}$$

Equation (A6) admits the following representation by potentials $\Phi = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}^{\mathrm{T}}$:

$$\Sigma_{1J} = -\Phi_{J,2} \quad \Sigma_{2J} = \Phi_{J,1}. \tag{A7}$$

Substituting eqn (A7) into eqn (A5), one obtains:

$$\begin{aligned} \mathbf{QU}_{,1} + \mathbf{RU}_{,2} &= -\mathbf{\Phi}_{,2} \\ \mathbf{R}^{\mathrm{T}}\mathbf{U}_{,1} + \mathbf{TU}_{,2} &= \mathbf{\Phi}_{,1}, \end{aligned} \tag{A8}$$

where

$$Q_{JM} = E_{1JM1}, \quad R_{JM} = E_{1JM2}, \quad T_{JM} = E_{2JM2}. \tag{A9}$$

It is noted that **Q** and **T** are symmetric matrices. Thus, eqn (A8) can be written in an eight-dimensional form as

$$\frac{\partial \mathbf{v}}{\partial x_2} = \mathbf{N} \frac{\partial \mathbf{v}}{\partial x_1}, \tag{A10}$$

where

$$\mathbf{v} = \begin{pmatrix} \mathbf{U} \\ \mathbf{\Phi} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{pmatrix} \tag{A11}$$

$$\begin{aligned} \mathbf{N}_1 &= -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^{\mathrm{T}}, \\ \mathbf{N}_3 &= \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} = \mathbf{N}_3^{\mathrm{T}}. \end{aligned} \tag{A12}$$

The inverse of the matrix **T** is obtained based on the following argument. From eqns (A4) and (A9), it follows that

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_e & \mathbf{e} \\ \mathbf{e}^{\mathrm{T}} & -\varepsilon_{22} \end{pmatrix}, \tag{A13}$$

where the 3×3 upper left corner of the matrix, $\mathbf{T}_e$, is formed from the part of the elastic tensor which is positive definite (Ting, 1986); the lower right corner is contributed from the dielectric tensor, a negative scalar; and **e** is formed by proper components of the piezoelectric tensor. It is found that

$$\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{T}_e^{-1}(\mathbf{I} + q\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{T}_e^{-1}) & -q\mathbf{T}_e^{-1}\mathbf{e} \\ -q\mathbf{e}^{\mathrm{T}}\mathbf{T}_e^{-1} & q \end{pmatrix}, \tag{A14}$$

where, by using the positive definiteness of $\mathbf{T}_e$,

$$q = \frac{1}{-\varepsilon_{22} - \mathbf{e}^{\mathrm{T}}\mathbf{T}_e^{-1}\mathbf{e}} < 0. \tag{A15}$$

In order to diagonalize eqn (A10), the following linear eigenvalue problem is considered:

$$\mathbf{N}\xi = p\xi. \tag{A16}$$

The eigenvalue is obtained by solving:

$$\det(\mathbf{N}-p\mathbf{I}) = 0. \tag{A17}$$

The explicit form of eqn (A17) is the same as eqn (9). The eigenvectors lead to the matrices **A** and **B** in Section 2.

APPENDIX B: THE GREEN'S FUNCTION

The Green's function solution, eqn (26), is in a one-complex-variable form which is adopted in the text to bypass the complexity caused by four complex variables (Suo *et al.*, 1992). The final solution has to be converted from this one-complex-variable form into the four-complex-variable representation. The conversion is done by the following procedure.

Without losing generality, let us consider a concentrated force acting on the free surface. The column **T** in eqn (26) is written explicitly as:

$$\mathbf{T} = (0, P, 0, 0)^{\mathrm{T}}. \tag{B1}$$

Noting eqn (21a), we have

$$\mathbf{f}'(z) = \mathbf{B}^{-1}\begin{Bmatrix} 0 \\ -P/2\pi i z \\ 0 \\ 0 \end{Bmatrix}, \tag{B2}$$

where

$$\mathbf{B}^{-1} = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} & b'_{14} \\ b'_{21} & b'_{22} & b'_{23} & b'_{24} \\ b'_{31} & b'_{32} & b'_{33} & b'_{34} \\ b'_{41} & b'_{42} & b'_{43} & b'_{44} \end{bmatrix}$$

is the inverse matrix of **B**. In terms of the four complex variables, we have

$$\mathbf{f}'(z_1, z_2, z_3, z_4) = -\frac{P}{2\pi i}\begin{Bmatrix} b'_{12}/z_1 \\ b'_{22}/z_2 \\ b'_{32}/z_3 \\ b'_{42}/z_4 \end{Bmatrix}, \tag{B3}$$

where $z_I (I = 1, 2, 3, 4)$ are given by eqn (13).

Furthermore, the traction–induction will be calculated from

$$\mathbf{h}(z_1, z_2, z_3, z_4) = \mathbf{B}\mathbf{f}'(z_1, z_2, z_3, z_4) = -\frac{P}{2\pi i}\begin{Bmatrix} \sum_{k=1}^{4} b_{1k}b'_{k2}/z_k \\ \sum_{k=1}^{4} b_{2k}b'_{k2}/z_k \\ \sum_{k=1}^{4} b_{3k}b'_{k2}/z_k \\ \sum_{k=1}^{4} b_{4k}b'_{k2}/z_k \end{Bmatrix}. \tag{B4}$$